Mostly in this class we want to solve DE analytically – not graphically

MAT 219 goal: Formulas for solutions

→ Explicit formulas $y = \dots$

→ Implicit formulas $F(x,y) = 0$

Plan: Divide DE into different types and develop a method for each type.

§2.2 Separable DE ($1^{st}$ order)

Separable DE can have variables "separated"

EX:

- $y' = \frac{N(x)}{M(y)}$ $\qquad$ $y' = \frac{3x^2 - x^3}{4 + y^3}$

- $M(y)\, y' = N(x)$ $\qquad$ $(4+y^3)\, y' = (3x^2 - x^3)$

- $M(y)\, dy = N(x)\, dx$ $\qquad$ $(4+y^3)\, dy = (3x^2 - x^3)\, dx$

(Recall $y' = \frac{dy}{dx}$)

Solution: Integrate the x & y functions

- $y' = \frac{N(x)}{M(y)}$

$M(y)\, y' = N(x)$

$\int M(y)\, y'\, dx = \int N(x)\, dx$

Note $y = y(x)$

$dy = y'\, dx$

- $\int M(y)\, dy = \int N(x)\, dx$ ← (Only write one const. of int)

- Solve for c.

EX: $y' = \frac{3x^2 - x^3}{4 + y^3}$

$(4+y^3)\, y' = (3x^2 - x^3)$

$\int 4 + y^3\, dy = \int 3x^2 - x^3\, dx$

$\left(4y + \frac{1}{4} y^4 = x^3 - \frac{1}{4} x^4 + C\right) \cdot 4$

$$\boxed{y^4 + x^4 + 16y - 4x^3 = C}$$

IVP $y' = \frac{3x^2 - x^3}{4 + y^3}$ w/ $y(0) = 1$

plug in $\begin{matrix} x = 0 \\ y = 1 \end{matrix}$ & solve for c.

## More examples.

**EX** $y' = \frac{x^2}{1-y^2}$ with $y(0)=2$

General solution:

$$(1-y^2)y' = x^2$$

$$\int 1-y^2\,dy = \int x^2\,dx$$

$$\left(y - \tfrac{1}{3}y^3 = \tfrac{1}{3}x^3 + C\right)\cdot 3$$

$$\boxed{3y - y^3 - x^3 = C}$$

IVP solution:

plug in $x=0$, $y=2$ $\qquad y(0)=2$

$$3(2) - (2)^3 - (0)^3 = C$$

$$-2 = C$$

$$\boxed{3y - y^3 - x^3 = -2}$$

**EX** $y' = \frac{y}{x}$

$$\int \tfrac{1}{y}\,dy = \int \tfrac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + C$$

$$|y| = C|x|$$

$$\boxed{\left|\frac{y}{x}\right| = C}$$

**EX** $y' = \frac{x^2}{y(1+x^3)}$ with $y(0) = 1$

Gen soln:

$$y\,y' = \frac{x^2}{1+x^3}$$

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx \qquad u = 1+x^3,\ du = 3x^2\,dx$$

$$\tfrac{1}{2}y^2 = \tfrac{1}{3}\ln|1+x^3| + C$$

$$\boxed{\tfrac{1}{2}y^2 - \tfrac{1}{3}\ln|1+x^3| = C}$$

IVP soln:

plug in $x=0$, $y=1$ $\qquad y(0)=1$

$$\tfrac{1}{2}\cdot(1)^2 - \tfrac{1}{3}\ln|1+0^3| = C$$

$$\tfrac{1}{2} = C$$

$$\boxed{\tfrac{1}{2}y^2 - \tfrac{1}{3}\ln|1+x^3| = \tfrac{1}{2}}$$

**EX:** $y' = xy + y$ with $y(2) = 1$

Gen soln:

$$y' = (x+1)y$$

$$\tfrac{1}{y}\,y' = (x+1)$$

$$\int \tfrac{1}{y}\,dy = \int x+1\,dx$$

$$\ln y = \tfrac{1}{2}x^2 + x + C$$

$$\boxed{\ln y - \tfrac{1}{2}x^2 - x = C}$$

IVP soln: etc...

Being separable is very special.

→ With some extra work we can change some DE to become separable

## Homogeneous DE

Total Degree of a term is sum of x & y powers

→ $x^n y^m$ has total degree $n+m$

$x^2y^3 \rightsquigarrow$ total degree $2+3=5$

$xy^2 \rightsquigarrow$ total degree $1+2=3$

$x^3 \rightsquigarrow$ total degree $3+0=3$

Homogeneous DE is $y' = \dfrac{p(x,y)}{q(x,y)}$

where all terms have same total degree

EX: $y' = \dfrac{x^2+xy+y^2}{x^2}$ Homogeneous.

$y' = x^2 + xy$ Not Homogeneous.

$y' = \dfrac{x^2+xy}{y}$ Not Homogeneous.

To solve homogeneous DE:

① Divide p & q by $x^{\text{total deg.}}$

② Substitute: $y/x = v$

$y' = v + xv'$

→ DE is now separable in v & x!

EX: $y' = \dfrac{x^2+xy+y^2}{x^2}$ ← total degree is 2

$y' = \dfrac{(x^2+xy+y^2)}{x^2} \dfrac{(1/x^2)}{(1/x^2)}$

$y' = \dfrac{1 + y/x + (y/x)^2}{1}$

$x \rightsquigarrow 1$

$y \rightsquigarrow v$

$y' \rightsquigarrow v + xv'$

$v + xv' = 1 + v + v^2$

$v' = \dfrac{1+v^2}{x}$

$\int \frac{1}{1+v^2}\,dv = \int 1/x \, dx$

$\arctan(v) = \ln|x| + c$

$$\boxed{\arctan(y/x) - \ln|x| = c}$$

7

**EX:** $y' = \dfrac{3y^2 - x^2}{2xy}$

$$y' = \frac{(3y^2 - x^2)}{2xy}\frac{(1/x^2)}{(1/x^2)}$$

$y/x = v$

$y' = v + xv'$

$$v + xv' = \frac{3v^2 - 1}{2v} - \frac{2v^2}{2v}$$

$$xv' = \frac{v^2 - 1}{2v}$$

$$\int \frac{2v}{v^2-1}\,dv = \int 1/x\,dx$$

$$\ln|v^2 - 1| = \ln|x| + c$$

$$|v^2 - 1| = c\,|x|$$

$$\left|\frac{y^2/x^2 - 1}{x}\right| = c$$

$$\boxed{\left|\frac{y^2 - x^2}{x^3}\right| = c}$$

**EX:** $y' = \dfrac{xy^2 + y^3}{x^3 + x^2y + xy^2}$

total degree is 3 for all terms

Homogeneous (total degree = 3)

$$y' = \frac{(xy^2 + y^3)}{(x^3 + x^2y + xy^2)}\frac{(1/x^3)}{(1/x^3)}$$

$$y' = \frac{y^2/x^2 + y^3/x^3}{1 + y/x + y^2/x^2}$$

Substitute:

$y/x = v$

$y' = v + xv'$

$$v + xv' = \frac{v^2 + v^3}{1 + v + v^2}$$

$$xv' = \frac{-v}{1 + v + v^2}$$

Separate variables and integrate

$$\int \frac{1 + v + v^2}{-v}\,dv = \int 1/x\,dx$$

$$-\ln|v| - v - \tfrac{1}{2}v^2 = \ln|x| + C$$

$$\boxed{+\ln|y/x| + y/x + \tfrac{1}{2}\,y^2/x^2 + \ln|x| = C}$$